Combinatorics

Compensatory Examination

Instructions: All questions carry equal marks.

1. Define a $t-(v,k,\lambda)$ design. Prove that there exists a $2-(15,3,1)$ design.

2. Prove that in any non-trivial $t-(v,k,1)$ design, we must have
$$v \geq (t+1)(k+1-t).$$

3. Prove that in a Steiner system $S(2,k,v)$ with $b > v$, we must have $v \geq k^2$. Further prove that, in case $v = k^2$, the set of blocks can be partitioned into $k+1$ "parallel classes" of k blocks such that the blocks in a given parallel class are pairwise disjoint. Such a Steiner system is called an *affine plane*.

4. Let a, d be natural numbers. If $b = \lceil \frac{a}{d} \rceil$ or $\lfloor \frac{a}{d} \rfloor$, then prove that
$$\lceil \frac{a-b}{d-1} \rceil \leq \lceil \frac{a}{d} \rceil \quad \text{and} \quad \lfloor \frac{a-b}{d-1} \rfloor \geq \lfloor \frac{a}{d} \rfloor.$$

5. An $S(2, n+1, n^2+n+1)$ Steiner system is called a *projective plane* of order n. Suppose P is a projective plane of order n, and L is a block. Show that if L and all the points on it are removed, then the resulting system of points and blocks is an affine plane (as defined above).